I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; CHARLES C. CROSS, Whaleyville, Va.; and H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Take $BC=a$, $AC=b$, and $AB=c$, and put the medians equal, respectively, m_a, m_b, and m_c. Then

$$m_a=\tfrac{1}{2}\sqrt{[2(b^2+c^2)-a^2]},$$
$$m_b=\tfrac{1}{2}\sqrt{[2(a^2+c^2)-b^2]},$$
$$m_c=\tfrac{1}{2}\sqrt{[2(a^2+b^2)-c^2]}.$$

But the medians of a triangle intersect at a common point two-thirds of the distance from the vertex to the middle of the opposite side. Whence

$$GA^2=(\tfrac{2}{3}m_a)^2=\tfrac{2}{9}b^2+\tfrac{2}{9}c^2-\tfrac{1}{9}a^2,$$
$$GB^2=(\tfrac{2}{3}m_b)^2=\tfrac{2}{9}a^2+\tfrac{2}{9}c^2-\tfrac{1}{9}b^2,$$
$$GC^2=(\tfrac{2}{3}m_c)^2=\tfrac{2}{9}a^2+\tfrac{2}{9}b^2-\tfrac{1}{9}c^2.$$

$\therefore$ $GA^2+GB^2+GC^2=\frac{1}{3}(a^2+b^2+c^2)$, and $3(GA^2+GB^2+GC^2)=a^2+b^2+c^2=BC^2+AC^2+AB^2$.

II. Solution by J. W. YOUNG, Fellow and Assistant, Ohio State University, Columbus, O.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the triangle, AM, BN, and CL the medians, and G the intersection of the medians. Then by a well-known theorem,

$$AC^2+CB^2=2CL^2+2.AL^2=2CL^2+\tfrac{1}{2}AB^2.$$

Similarly,

$$CB^2+BA^2=2BN^2+\tfrac{1}{2}AB^2,$$
$$BA^2+AC^2=2AM^2+\tfrac{1}{2}CB^2.$$

Adding and dividing by 2, we have

$$AC^2+CB^2+BA^2=CL^2+BN^2+AM^2+\tfrac{1}{4}(AB^2+AC^2+CB^2),$$
$$\text{or } AC^2+CB^2+BA^2+\tfrac{4}{3}(CL^2+BN^2+AM^2),$$

(since G divides medians in the ratio 2:1)

$$=\tfrac{4}{3}(\tfrac{9}{4}CG^2+\tfrac{9}{4}BG^2+\tfrac{9}{4}AG^2)=3(CG^2+BG^2+AG^2.$$

The same propositions can very easily be proven analytically.

Solved in a similar manner by *COOPER D. SCHMITT*, *G. B. M. ZERR*, *WALTER H. DRANE*, and *CHAS. C. CROSS*.

124. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Every conic that passes through all the foci of a conic is a rectangular hyperbola. [From Charlotte A. Scott's *Modern Analytical Geometry*.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If $u=0 \ldots\ldots(1)$ be the general equation of the second degree, we have the well-known equations

$$Fx+Gy-H=Cxy \ldots\ldots(2),$$

$$2Gx-2Fy-A+B=C(x^2-y^2) \ldots\ldots(3),$$

for the foci, A, B, etc., being, as usual, the various minors of

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Either (2) or (3) is the rectangular hyperbola.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

"If through each of the two imaginary points at infinity on any circle two tangents be drawn to the conic, these tangents will form a quadrilateral whose vertices will be the real and imaginary foci of the conic."

Both Puckle and Salmon solve this problem. Salmon's proof expanded is as follows:

Find the condition that $x-x'+(y-y')\sqrt{(-1)}=0$ should touch the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The condition that $mx+ny+p=0$ touches the same conic is

$$(bc-f^2)m^2+(ca-g^2)n^2+(ab-h^2)p^2+2(gh-af)pn+2(hf-bg)pm$$
$$+2(fg-ch)mn=0,$$
$$\text{or } Am^2+Bn^2+Cp^2+2Fpn+2Gpm+2Hmn=0.$$

Now $m=1$, $n=\sqrt{-1}$, $p=-x'-y'\sqrt{-1}$ or dropping the accents $p=-(x+y\sqrt{-1})$.

$\therefore A-B+Cx^2-Cy^2+2Cxy\sqrt{(-1)}-2Fx\sqrt{(-1)}+2Fy-2Gx-2Gy\sqrt{(-1)}$ $+2H\sqrt{(-1)}=0$.

$\therefore C(x^2-y^2)+2Fy-2Gx+A-B=0 \ldots\ldots(1)$,

and $2Cxy-2Fx-2Gy+2H=0$, or

$$Cxy-Fx-Gy+H=0 \;\ldots\ldots(2).$$

The intersections of (1) and (2), both of which are rectangular hyperbolas, determine the foci.